Order-restricted Bayesian Inference and Optimal Designs for the Simple Step-stress Accelerated Life Tests under Progressive Type-I Censoring

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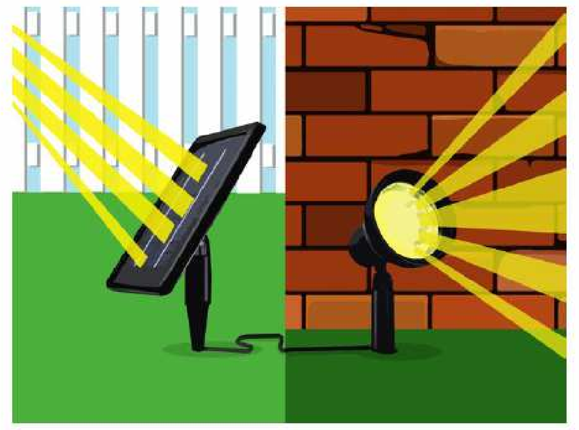
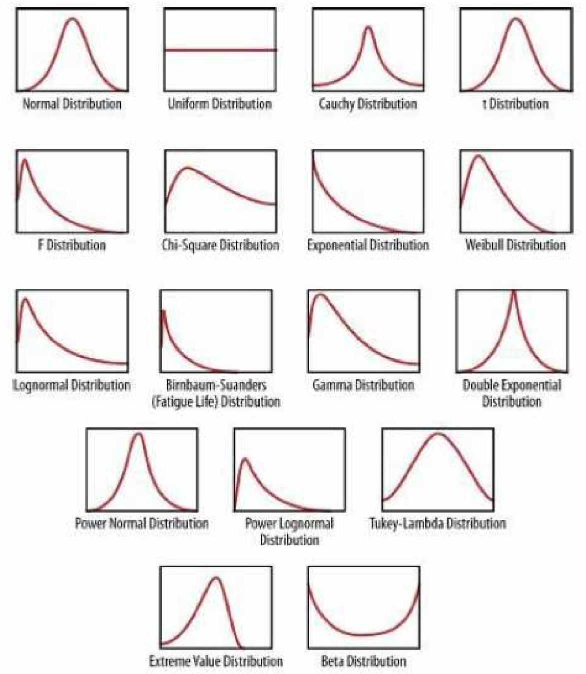
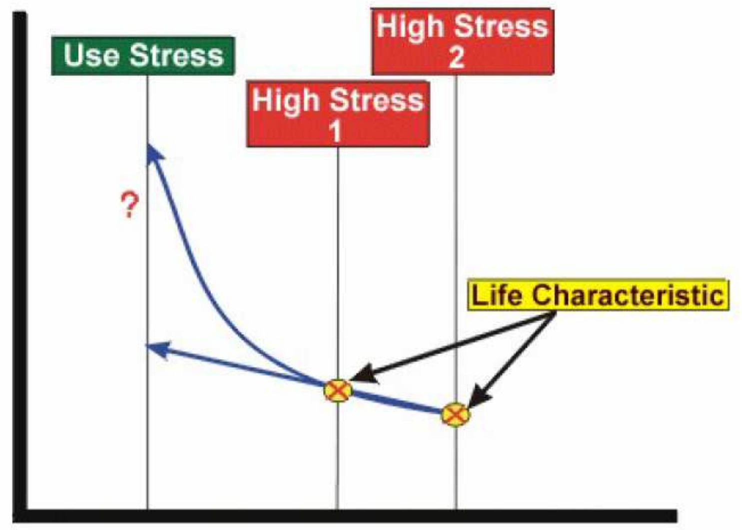
Life Testing

We want to determine the life expectancy (distribution) of some product or device, let's say a solar lighting device. What can we do?





Simple Step-Stress Accelerated Life Testing



Bayesian Statistics

When frequentist approaches become computationally challenging or rely too heavily on asymptotic properties, Bayesian techniques can be a useful alternative.

- Allows for the use of "prior" information, $\pi(\boldsymbol{\lambda})$.
- Conjugate priors can be used to help avoid the need for expensive MCMC computations.
- Inference can be made using marginal posterior distributions, e.g.) inference for λ_1 using $\pi(\lambda_1|\boldsymbol{t})$.
- Designs can be more attractive if the sample size is small.

Bayes' Theorem

$$\pi(\boldsymbol{\lambda}|\boldsymbol{t}) = \frac{L(\boldsymbol{\lambda}|\boldsymbol{t})\pi(\boldsymbol{\lambda})}{\int L(\boldsymbol{\lambda}|\boldsymbol{t})\pi(\boldsymbol{\lambda})d\boldsymbol{\lambda}}$$

The goal of this presentation is to introduce a computationally more appealing Bayesian approach to order-restricted inference for a simple step-stress accelerated life testing and its design optimization.

Likelihood Function

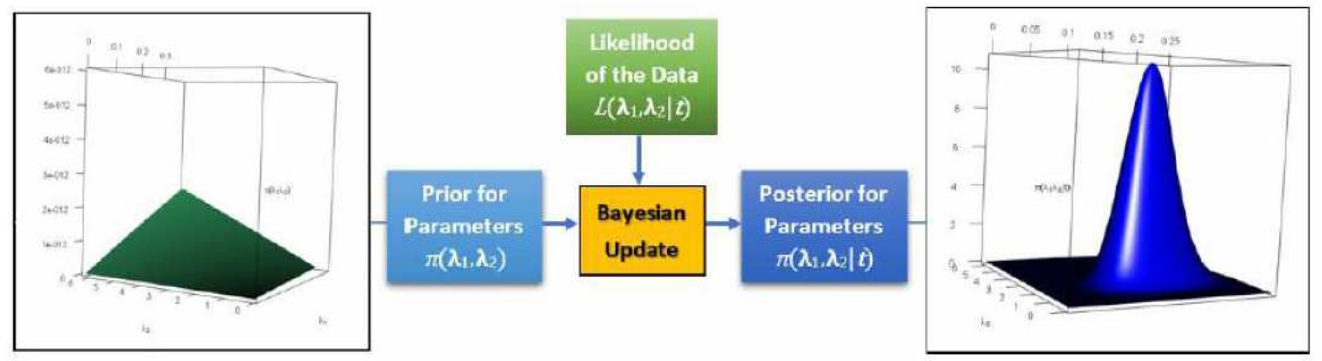
Assuming that a cumulative exposure model is appropriate and that the life distribution of a test unit is exponential at any level of stress, the likelihood function is:

$$L(\lambda_1, \lambda_2|\boldsymbol{t}) = \lambda_1^{n_1}\lambda_2^{n_2}\exp(-\lambda_1 U_1 - \lambda_2 U_2) \qquad (1)$$

Here, n_i are the number of units that failed at the respective stress level x_i, i.e.) the number of failures observed in the time interval (τ_{i-1}, τ_i), and $U_i = \sum_{j=1}^{n_i}(t_{i,j} - \tau_{i-1}) + (N_i - n_i)(\tau_i - \tau_{i-1})$. N_i are the number of units entering at the respective stress level which depend on the the censoring proportion π^*.

Bayesian Framework

- Likelihood is as seen in Equation 1.
- Prior is a 3-parameter gamma distribution.
- Posterior results in a tractable expression that is a mixture of gamma densities.



Prior Distribution: 3-parameter Gamma

$$\pi(\lambda_1) = \frac{\gamma_1^{\alpha_1}}{(\alpha_1 - 1)!}(\lambda_1)^{\alpha_1 - 1}\exp(-\gamma_1\lambda_1)$$

$$\pi(\lambda_2|\lambda_1) = \frac{\gamma_2^{\alpha_2}}{(\alpha_2 - 1)!}(\lambda_2 - \lambda_1)^{\alpha_2 - 1}\exp(-\gamma_2(\lambda_2 - \lambda_1))$$

$$\alpha_i \in \{1, 2, 3, \dots\} \text{ and } \gamma_i > 0 \qquad i = 1, 2.$$

- Shift parameter λ_1 imposes the order restriction.
- γ_1 and γ_2 are the rate hyperparameters.
- α_1 and α_2 are the shape hyperparameters.

Cumulative Exposure Distribution

Solar Lighting device data from Han and Kundu (2015) was used.

$$G(t) = \begin{cases} 1 - \exp(-\lambda_1 * t) & \text{if } 0 < t < \tau_1 \\ 1 - \exp(-\lambda_2 * (t - (1 - \lambda_1/\lambda_2) * \tau_1)) & \text{if } \tau_1 \le t < \infty \end{cases}$$

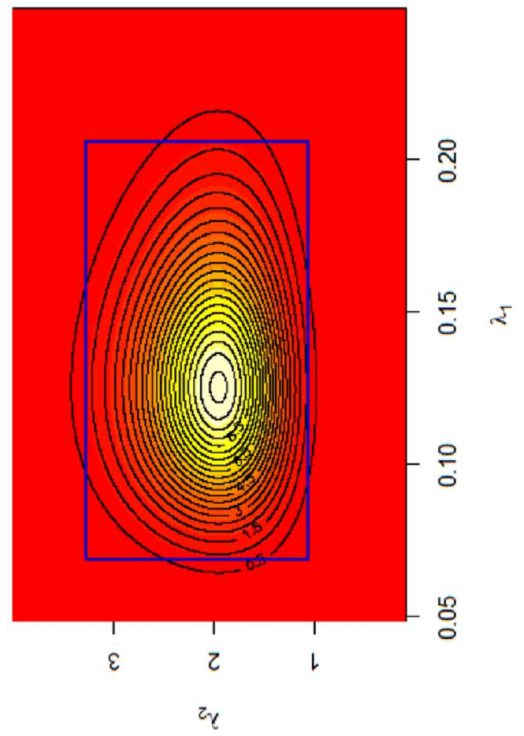
Solar Lighting Inference

n $= 35$, $\alpha_1 = \alpha_2 = 2$, $\gamma_1 = \gamma_2 = 0.001$

λ_1			
Mean	Median	Mode	Variance
0.132	0.130	0.125	0.001

λ_2			
Mean	Median	Mode	Variance
2.083	2.042	1.961	0.253

The covariance was found to be 7.05686e-05.



Monte Carlo Simulations

Parameter and hyperparameter settings:

- $\lambda_1 = 1.1052$, $\lambda_2 = 2.7183$
- $\alpha_1 = \alpha_2 = 2$
- $\gamma_1 = \gamma_2 = 0.001$ and then $\gamma_1 = \gamma_2 = 0.0001$

The selections made for λ_1 and λ_2 were motivated by the desire to follow choices made for related frequentist work of Han and Bai (2020).

Using 1,000 simulations with $n = 24$ and then repeated for $n = 48$, estimators for the parameters of the posterior in Equation 2 were approximated by Monte Carlo simulations. Total test duration choices were $\tau = 0.9$, $\tau = 1.2$ and $\tau = 1.5$.

Given the progressive Type-I censoring scheme, choices for the proportion of surviving units to censor after the first level were chosen as $\pi = 0\%$, $\pi = 10\%$ and $\pi = 20\%$.

Simulation: n $= 24$, $\alpha_1 = \alpha_2 = 2$, $\gamma_1 = \gamma_2 = \gamma$

			λ_1				λ_2				
γ	τ	π^*	Mean	Median	Mode	Variance	Mean	Median	Mode	Variance	Covariance
0.0001	0.9	0%	1.240	1.207	1.140	0.131	3.681	3.587	3.395	0.993	0.042
		10%	1.242	1.209	1.142	0.132	3.788	3.682	3.468	1.143	0.044
		20%	1.245	1.211	1.143	0.133	3.913	3.794	3.552	1.363	0.047
	1.2	0%	1.220	1.192	1.136	0.108	3.766	3.664	3.458	1.134	0.034
		10%	1.221	1.194	1.137	0.108	3.871	3.758	3.528	1.358	0.036
		20%	1.223	1.195	1.138	0.109	4.008	3.880	3.619	1.597	0.038
	1.5	0%	1.210	1.186	1.136	0.093	3.887	3.772	3.539	1.420	0.031
		10%	1.212	1.188	1.138	0.094	4.035	3.906	3.644	1.711	0.032
		20%	1.214	1.189	1.139	0.095	4.182	4.037	3.742	2.023	0.035
0.0010	0.9	0%	1.240	1.207	1.140	0.131	3.680	3.586	3.394	0.993	0.042
		10%	1.242	1.209	1.142	0.132	3.787	3.681	3.467	1.142	0.044
		20%	1.245	1.211	1.143	0.133	3.912	3.793	3.551	1.362	0.047
	1.2	0%	1.220	1.192	1.136	0.108	3.765	3.663	3.457	1.133	0.034
		10%	1.221	1.194	1.137	0.108	3.870	3.757	3.527	1.356	0.036
		20%	1.223	1.195	1.138	0.109	4.006	3.878	3.618	1.596	0.038
	1.5	0%	1.210	1.186	1.136	0.093	3.886	3.771	3.538	1.418	0.031
		10%	1.212	1.188	1.138	0.094	4.033	3.905	3.642	1.709	0.032
		20%	1.214	1.189	1.139	0.095	4.181	4.036	3.740	2.020	0.035

Design Optimization

Using equal step durations, the optimal value for the total test duration was obtained under the information-theoretic design criterion H as well as various criteria $D/C/A/M/E$ based on the posterior variance-covariance matrix of $\boldsymbol{\lambda}$ (or $\boldsymbol{\beta}$ based on a linear link).

Design Utilities

- H-optimal design maximizes the expected information gain based on the posterior entropy.
- D-optimal design maximizes the expected determinant of the inverse of the posterior variance-covariance matrix.
- C-optimal design maximizes the expected reciprocal of the posterior variance at normal operating conditions.
- A-optimal design maximizes the expected reciprocal of trace of the posterior variance-covariance matrix.
- M-optimal design maximizes the expected reciprocal of the maximum posterior variance.
- E-optimal design maximizes the expected minimum eigenvalue of the inverse of the posterior variance-covariance matrix.

		H-optimality		D-optimality		A-optimality		M-optimality		E-optimality	
n	π^*	Δ^*	U^*	Δ^*	U^*	Δ^*	U^*	Δ^*	U^*	Δ^*	U^*
24	0%	3.03157	1.54094	1.68886	0.07531	0.94991	0.65499	0.80339	0.54297	0.80462	0.55817
	10%	3.14477	1.51972	2.06914	0.07753	0.97278	0.68724	0.81491	0.57791	0.87029	0.59303
	20%	3.31735	1.49254	2.26275	0.07989	0.92093	0.72582	0.79772	0.61535	0.85347	0.63023
48	0%	2.35213	2.04859	1.09102	0.03033	0.87307	0.39953	0.72381	0.33424	0.74188	0.34037
	10%	2.35499	2.01628	1.11958	0.03202	0.84862	0.42509	0.76127	0.36039	0.77342	0.36611
	20%	2.55947	1.98132	1.27335	0.03429	0.81280	0.45669	0.74901	0.39041	0.76193	0.39650

Conclusions

- Using a 3-parameter gamma distribution as a conditional prior, we have performed Bayesian estimation and design optimization for progressively Type-I censored simple SSALTs under continuous inspections assuming that the lifetimes are exponential and that a cumulative exposure model holds.
- This prior ensures that the failure rates increase as the stress level increases.
- This prior leads to a tractable joint posterior distribution, which is a mixture of gamma densities.

Future Directions

- Extending this framework to the interval monitoring setting
- Exploring different censoring schemes